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IRON ARCHES.

THE

PRACTICAL THEORY

OF

THE CONTINUOUS ARCH.

BY

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IRON ARCHES.

THE practical theory of arches has probably made less progress than any other branch of engineering knowledge. There are in common use two forms of arch, entirely distinct in their mechanical arrangement and source of strength, viz., the voussoir arch built of separate stones, and the continuous arch of wood or iron. The first of these has, indeed, received much attention from mathematicians, but from the unsatisfactory and indeterminate character of the problem of the arch, when the friction between the stones is considered, the theory has in general been confined to the various forms of equilibrated arches, which stand in virtue of the adjustment of the loads on each voussoir, without calling into play any friction forces between the voussoirs. It is, however, manifest that, for a heavy movable load, such as almost all arches are liable to, any theory of the arch which neglects friction is very far from complete, and such is the power of this friction to secure the stones of an arch, that in practice the theoretical forms are rarely considered, and the shape of the arch is decided upon in accordance with points of appearance or convenience. The friction, indeed, never reaches its limit in practice, and arches fail, not by the voussoirs slipping upon one another, but by the resultant line of thrust

falling too near to the outside of the arch. Since, then, the friction never reaches its limit, it is impossible to say what its value may be for a given disposition of load, and hence the indeterminate nature of the problem, as will be pointed out hereafter.

For the continuous arch there is, so far as the writer is aware, no published investigation of a practical character. This form of arch is mainly due to the rapid progress of construction in wrought iron, and from its strength and security it has become the favourite form of arch for bridges of large span. It is, therefore, of great importance to investigate, however approximately, the strains on such arches, and there is less difficulty and uncertainty in doing so for a continuous arch of wrought iron than for a voussoir arch, because in general the depth of the ribs of a continuous arch is small compared with the radius of the arch, and this circumstance offers great facilities for approximation. The main object of the present communication is to investigate the strains on a continuous iron arch.

It will, however, be proper, first, to examine into the conditions of equilibrium of a voussoir arch, in order to show the indeterminate character of the problem, as also to point out the different nature of the forces which the two forms of arch are able to bring into play to resist rupture, and on which they rely for their stability. This may be done shortly as follows:

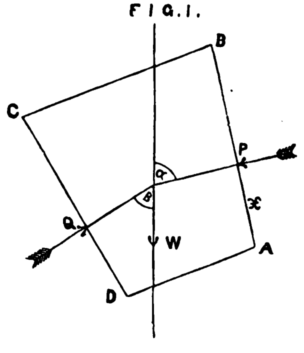
Consideration of an equilibrated arch, i.e., an arch constructed of voussoirs whose weights are so adjusted that every voussoir is in equilibrium without calling into action any force of friction between adjacent voussoirs.

(It will be found proved in works on mechanics that this will be the case when the weights of the voussoirs are proportional to the difference of the tangents of the angles which their joints make with the vertical.)

In this case there will be no forces acting at the joints of a voussoir, but the pressures of the adjacent voussoirs: these will all be normal to the joints at which they act, and the resultant of such forces at each joint will, therefore, also be normal to the joint.

Let, then, A B C D, Fig. 1, be a stone of an equilibrated arch. This stone is in equilibrium under the action of three forces, viz.; the pressures at the joints collected into their resultants, P and Q, and the weight, W, of the stone. These three forces must, therefore, meet in a point in the vertical through the centre of gravity of the stone, as shown in the figure, and since the position of the stone is supposed known, the angles α and β, which the directions of the side forces make with the vertical, are also known. The force P will be known in terms of the weights of the voussoirs between the

FIG. 1.



crown of the arch and the stone in question, and may be assumed to act at a distance x from A, measured along the joint A B. Then resolving the forces vertically and horizontally, and taking moments about the point D, we have the three following equations:

$$Q \cos \beta - P \cos \alpha - W = 0$$
$$Q \sin \beta - P \sin \alpha = 0$$
$$f(x, Q, \&c.) = 0$$

where f (x, Q, &c.) means an expression involving the quantities, x, Q, &c.

All the quantities involved in these three equations are known, except Q and x. Consequently there are three equations, and only two unknown quantities, and it follows that one of the equations, evidently the equation of moments, is a consequence of the other two. Since, then, the equilibrium of the voussoir is sufficiently provided for without using the equation of moments, which alone involves the point of application of P, it follows that for the equilibrium of the stone it is immaterial at what point of the stone the force P is supposed to act, and x may have any value that the depth of the stone will allow. Thus as regards the equilibrium of a single stone, the position of the resultant line of thrust is wholly indeterminate.

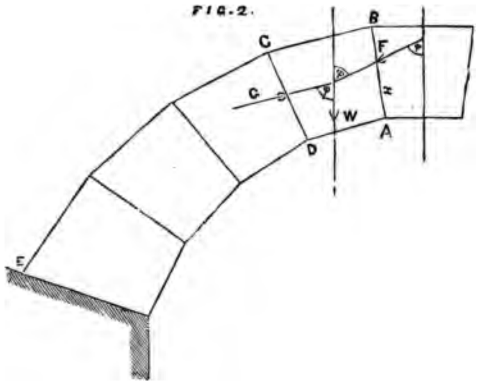
When, however, the arch, of which A B C D is a single voussoir, is considered collectively, it will be seen that the range of variation of the resultant line of thrust is reduced within more moderate limits. For the resultant line of thrust must not pass out of the arch either above or below, and this condition will reduce the range of x, more or less according to the shape of the arch and the depth of the stones. Nevertheless there will in general remain a considerable range of variation, and the exact position of the resultant line of thrust is an indeterminate problem.

Consideration of an arch with friction between the voussoirs.

Suppose that an equilibrated arch as above considered has its equilibrium destroyed by the addition of a load upon the crown of the arch. Then the pressures at all the joints throughout the arch will be increased, and unless the friction between the stones were to come into play, each voussoir, as A B C D, would be thrust out of the arch upwards. At every joint, however, a force of friction will be called into action to oppose this tendency, which, together with the normal pressure

between the adjacent stones, will be represented by a resultant force inclined to the joints at an angle dependent on the amount of friction called into play. Let F and G represent these resultant forces in the case of the voussoir in the diagram, Fig. 2, inclined to the vertical at unknown angles φ and ψ,

FIG. 2.



and let the force F be supposed to act at a point distant x from A. Then, as before, the stone is in equilibrium under the action of the three forces, F, G, and W, and the conditions of equilibrium give rise to the following equations:

$$G \cos \psi - F \cos \varphi - W = 0$$
$$G \sin \psi - F \sin \varphi = 0$$
$$f(x, G, F, \varphi, \psi, \&c.) = 0$$

where $f(x, G, F, \varphi, \psi, \&c.)$ means an expression involving the quantities x, G, F, φ, ψ, &c.: F is known in terms of the angle φ and the weights of the voussoirs, &c., between the crown of the arch and the stone in question.

Here, then, we have three equations and four unknown quantities, viz., x, G, φ, and ψ: if we eliminate two of them, as G and ψ, we obtain an equation of condition between x and φ, which will be satisfied by an infinite number of pairs of values of x and φ. Now φ depends upon the amount of friction acting along the joint A B, and the equation of con-

dition between x and φ shows that the amount of this friction may vary, provided that x undergoes a corresponding variation, as defined by the equation. Thus in this case also x has a range of variation, within the limits of which equilibrium may subsist, and the precise position of the resultant line of thrust is as before, an indeterminate problem.

It is now possible to investigate the specific points of difference between the arch constructed of unconnected voussoirs and a continuous arch as constructed in wood and iron. This will be best demonstrated by considering the forces which the two forms of arch are able to bring into play to counteract the bending moment at any point of the arch. In the case of the voussoir arch above considered, let the portion A E, between the joint A B and the abutment, be supposed united in one rigid mass. Then the forces which act upon the mass A E are, (1) the thrust of the abutment; (2) the weight of the mass A E, acting through its centre of gravity; (3) the pressure at the joint A B. Let it be supposed that the mass A E is in equilibrium for a given disposition of load, when the force F acts at a distance (x) from A. Then the moments of the three forces about the point B balance one another, and the joint A B has no tendency to open either at A or at B. If now the disposition of load be altered, the forces will be altered and their moments about B will no longer balance one another, and there will be a resultant moment tending to make the joint open either at A or at B. Let it be supposed that the tendency of this resultant moment is to make the joint open at A. Then the point of application of the force F (which tends to open the joint at A) shifts nearer to B (as is permitted by the conditions of equilibrium stated in the last paragraph) so as to diminish the effect of its moment about B, and thus equilibrium is maintained. Finally, when the tendency to open at A is so great that it cannot be counteracted, except the point of application of the force F retires past B, and outside the arch, then the joint opens at A, and

the arch falls to pieces. From this is seen the great importance of depth for the stones of a voussoir arch. As regards the strain on the material of the arch this will be, of course, a simply compressive strain, distributed more or less according to the degree of elasticity which exists in the stone. It is certain that the edges of the voussoirs would crush or flake off long before the resultant line of thrust was driven to the outside surface of the arch; but the problem, as regards the strain on the materials, is, generally speaking, indeterminate, and the rules which have been established on the subject are entirely empirical.

In the case of the continuous arch it must be understood that there is acting at every point a bending moment and a thrust force. The effect of these two as regards the equilibrium of the mass A E will be represented by their resultant, which will be a single thrust force, acting at a point removed from the centre of the section, and precisely similar to the thrust force F of the voussoir arch; but as regards the strain on the materials the effect is far more definite than in the case of the voussoir arch. For a continuous arch can bring into play forces of tension as well as forces of compression, and the bending moment is met by an opposite moment about the neutral axis of the arch, which is supplied by the resistance of the materials in the manner of an ordinary plate-girder. Thus in this case the arch relies upon the strength of the materials to resist the bending moment of the forces, and the total effect of the forces upon the material of the arch will be ascertained by combining with this moment the thrust of the arch, which will be known in terms of the weights and other forces which act upon the arch. This force of thrust will clearly increase the strain upon that side of the arch which is in compression in consequence of the bending moment, and will relieve that side which is in tension, so that in general the arch would fail by compression; but in any case the stability of a continuous arch depends in a strictly

definite manner on the strength of the materials, and thus it becomes possible, by using reliable materials of great strength, such as cast and wrought iron, the action of which under tension and compression is well known, to cross a wide span with much less depth of rib than would be required by a voussoir arch.

There yet remains a force which does not enter into the consideration of the bending moment either in the case of the voussoir arch or the continuous arch, viz.: the force along the surface of the section, which in the voussoir arch takes the form of friction between the joints, as already described, and in the continuous arch takes the form of a shearing force, which can be expressed in terms of the thrust force and the inclination of the curve drawn through the successive positions of the resultant force to the section in question. This shearing force will in general have but a very slight effect on the security of an arch.

The following investigation of the bending moment at any point of an iron arch is due to the Astronomer-Royal, and although the results are given only for the ordinary case of a circular arch, yet the method is equally applicable to the case of more complicated arches. As a preliminary to the investigation, the conditions of breaking and bending of an iron bar are prescribed in the two following propositions. Throughout the investigation the unit of length is supposed to be 1 ft., and the unit of weight the weight of a cubic foot of iron.

PROPOSITION I. *To investigate the criterion of breaking of a bar, by bending forces whose angular moment is M, at any particular point of the bar.*

The bar will break when the rending force at the surface on the extended side is equal to the tension strength of iron. Call this (t), where (t) is to be expressed numerically by the number of feet in length of any bar whose weight will tear that bar asunder. Suppose the section of the bar to be a

parallelogram (there is no difficulty in making a more general supposition when necessary), (b) its whole depth in the direction in which it will bend (the plane of the arch), and (a) its thickness. And suppose the neutral point to be in the centre of the bar's depth. Measure (x) from the centre towards the stretching side. The surface of that part of the section which corresponds to δx is $a. \delta x$: the extension is $\frac{2x}{b} \times$ extension at the tearing surface; the tension force for surface 1 is $\frac{2x}{b} \times$ tearing force $= \frac{2tx}{b}$; the tension force for surface $a. \delta x$ is $\frac{2at}{b} x. \delta x$; its moment round the neutral point is $\frac{2at}{b} x^2. \delta x$; the sum of all the moments from the centre to x is $\frac{2at}{3b} x^3$. Taking this from $x = -\frac{b}{2}$ to $x = +\frac{b}{2}$ (which includes the thrusting effect of the compressed side) the entire sum of moments is $\frac{4at}{3b} \cdot \frac{b^3}{8} = \frac{ab^2t}{6}$. At the moment of breaking this must $=$ M, or $M = \frac{ab^2t}{6}$.

PROPOSITION II. *To investigate the bending produced in a bar by bending forces whose moment is M; no consideration of breaking being entertained.*

Suppose that a weight 1 hung to a bar whose section is 1 will extend the bar by $e \times$ bar's length (e is a very small fraction). Let (r) be the radius of the curvature which the application of the moment M produces in the neutral line. At the distance (x) from the neutral line the fibres are lengthened in the proportion $r : r+x$, or by the $\frac{x\text{th}}{r}$ part of their length. And since an extension $e \times$ fibres' length corresponds to weight 1 on section 1, therefore an extension

$\frac{x}{r} \times$ fibres' length corresponds to weight $\frac{x}{r.e}$ on section 1, and therefore to weight $\frac{x}{r.e}.\ a.\ \delta\, x$ on section $a.\ \delta\, x$. Therefore the force actually exerted by the part $a.\ \delta\, x$ in its stretched state is $\frac{a}{r.e}.\ x\ \delta\, x$, and its moment is $\frac{a}{r.e}.\ x^2.\ \delta\, x$; the sum of all the moments is $\frac{a}{3.r.e}\ x^3$: which from $x = -\frac{b}{2}$ to $x = +\frac{b}{2}$ becomes $\frac{2.a}{3.r.e}.\ \frac{b^3}{8} = \frac{a.b^3}{12.r.e}$. This sum of moments resisting the bending is equal to the moment producing bending $=$ M. Therefore $\frac{1}{r} = \frac{12.e}{a.b^3}.\ \mathrm{M}$. This $\frac{1}{r}$ may be taken as the measure of curvature.

In the following investigation, if the dimensions of the bar were supposed to vary, we must use the factor $\frac{12.e}{a.b^3}$; but if we consider the dimensions of the arch to be everywhere the same, we may put E for $\frac{12.e}{ab^3}$, and then $\frac{1}{r} = \mathrm{E.\,M}$.

The course of the investigations will be:

1. The arch will be supposed to be an arc of a circle, of uniform breadth and depth throughout, loaded with a weight on its crown. (Probably there will be no difficulty, though some complexity, in varying both these conditions.) It is assumed that the depth of the rib is small compared with the span of the arch.

2. It will be supposed that the feet of the arch are prevented from spreading asunder. (There is no difficulty in supposing the feet free to spread: but this is not the case in practice.)

3. At any point of the arch, the moment M will be investigated, including terms which depend on the horizontal forces

H, H, which prevent the feet of the arch from spreading asunder.

4. The impressed curvature at every part of the arch will therefore be found, including effects of H.

5. From these curvatures will be found the spread given to the feet of the arch, still involving H.

6. This spread will be made$=0$, whence H will be determined.

7. The value of H will be substituted in M (No. 3), and the explicit value of M at every point of the arch will be found.

The point of the arch will be found at which $\frac{6\,M}{a.b.^2t}$ (Prop. I.) is maximum: that point is the point most likely to break. If $\frac{6\,M}{a.b.^2t}$ is greater than 1, the arch will break there.

Proposition III. *To investigate the moment M at any point of the arch.*

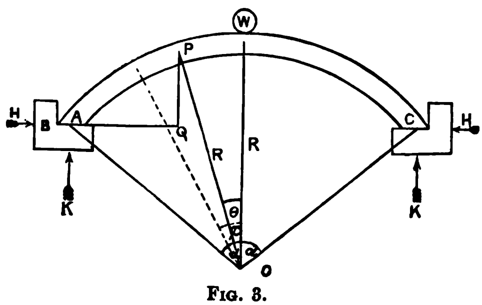


Fig. 3.

Let A C be the arch, and P the point at which the bending moment is required to be known. The position of P is defined by its radial distance R, from O the centre of the circle, and the angle θ, which this radius makes with the vertical. P Q is a vertical from P, and A Q a horizontal line meeting P Q in Q. (α) is the angle of the half arch, and $a\ b$ the sectional area of the rib.

The abutments are drawn in the rectangular state to show

the resolved actions of the supporting forces; but they would be exactly the same if the abutments were inclined.

It is supposed in the figure that the bearing of the arch is at the point A, at the moment of breaking. Strictly speaking it will be most probably at B: but in what follows this merely makes the difference that we ought to have taken a rather longer arch, so that the radius to the assumed bearing point would have been directed towards B.

It is supposed that the arch is originally planted without any strain, or in exactly the same form which it would have taken if laid sideways on a horizontal plane.

For estimation of the moment about P, it will be sufficient to take one side of P (inasmuch as that on the opposite side is necessarily equal to it). Take it on the side towards A; then the whole moment to produce bend at P consists of the following parts:

$\left.\begin{array}{l}\text{The weight of the portion from P to A}\\ \text{The force H}\end{array}\right\}$ in one direction, bending A downwards and inwards—and the force $K = R\,\alpha\,a\,b + \frac{W}{2}$, in the opposite direction, bending A upwards and outwards.

1. The portion from P to A.

The mass between φ and $\varphi + \delta\varphi$ is $a.\ b.\ R.\ \delta\varphi$; its horizontal distance from P is $R(\sin\varphi - \sin\theta)$; its moment round P is $a.\ b.\ R^2(\sin\varphi - \sin\theta)\,\delta\varphi$; the entire moment is $a.\ b.\ R^2(-\cos\varphi - \sin\theta.\ \varphi)$; taking this between the limits $\varphi = \theta$, $\varphi = \alpha$, the entire moment round P of the portion from P to A becomes $a.\ b.\ R^2(\cos\theta + \theta\sin\theta - \cos\alpha - \alpha\sin\theta)$.

2. The force H.

Its moment round $P = H \times PQ = H.R(\cos\theta - \cos\alpha)$.

3. The force $R\,\alpha\,a\,b + \frac{W}{2}$.

Its absolute moment is $\left(R\,\alpha\,a\,b + \frac{W}{2}\right) \times AQ =$

$\left(R\,\alpha\,a\,b + \frac{W}{2}\right) \times R\,(\sin\alpha - \sin\theta)$: but as this is in the opposite direction, it must be taken

$$\left(R\,\alpha\,a\,b + \frac{W}{2}\right) \times R\,(\sin\theta - \sin\alpha).$$

Hence the entire value of M at P is

$$(a\,b\,R^2 + H.R)\,(\cos\theta - \cos\alpha) + a\,b\,R^2\,(\theta - \alpha)\sin\theta$$
$$+\left(R^2\,a\,b\,\alpha + \frac{R\,W}{2}\right)(\sin\theta - \sin\alpha) =$$
$$(a\,b\,R^2 + H\,R)\,(\cos\theta - \cos\alpha)$$
$$+\ a\,b\,R^2\,(\theta\sin\theta - \alpha\sin\alpha) + \frac{R.W}{2}\,(\sin\theta - \sin\alpha).$$

PROPOSITION IV. *To find the impressed curvature at P.*

By Proposition II., the reciprocal of the radius of the circle into which straight fibres are bent is E × M. Or, the reciprocal of the radius of the circle into which straight fibres are bent, omitting the general factor E R, is

$$(a\,b\,R + H)\,(\cos\theta - \cos\alpha) + a\,b\,R\,(\theta\sin\theta - \alpha\sin\alpha)$$
$$+\ \frac{W}{2}\,(\sin\theta - \sin\alpha).$$

If then we consider two points on the neutral line whose curved distances from the crown of the arch are $R.\,\theta$, $R.\,(\theta + \delta\theta)$, the fibres at the second point are bent downwards with respect to their original direction in regard to those of the first by the angle

$$E.\,R^2.\delta\theta\left\{(a\,b\,R + H)\,(\cos\theta - \cos\alpha)\ +\right.$$
$$\left.a\,b\,R\,(\theta\sin\theta - \alpha\sin\alpha) + \frac{W}{2}\,(\sin\theta - \sin\alpha)\right\}$$

PROPOSITION V. *To find the spread of the foot A.*

(It will he convenient to estimate it positive in the inward direction.) Let the bend at the end of the last proposition be called $\Theta.\,\delta\,\theta$. Join A and P by a straight line. The bending will throw A through the space $A\,P \times \Theta.\,\delta\,\theta$, in the direction

perpendicular to A P. The resolved part of this inwards is

$$\text{A P} \times \Theta.\,\delta\,\theta \times \frac{\text{P Q}}{\text{P A}} = \text{P Q} \times \Theta.\,\delta\,\theta =$$

$$\text{R}.\,(\cos\theta - \cos\alpha).\quad \Theta.\,\delta\,\theta. = \text{E}.\,\text{R}.^3\,\delta\,\theta \times$$

$$\left\{\begin{array}{l} (a\,b\,\text{R} + \text{H})\,(\cos\theta - \cos\alpha)^2 \\ + a\,b\,\text{R} \times (\cos\theta - \cos\alpha) \times (\theta.\,\sin\theta - \alpha\sin\alpha) \\ + \dfrac{\text{W}}{2} \times (\cos\theta - \cos\alpha) \times (\sin\theta - \sin\alpha). \end{array}\right\}$$

This must be integrated from $\theta = 0$ to $\theta = \alpha$, to obtain the entire inwards spread of A, and without giving the details of the integration, the entire expression for the spread of A inwards, omitting the factor E R^3, appears as

$$(a\,b\,\text{R} + \text{H}) \times \{-\tfrac{3}{2}\sin\alpha\cos\alpha +$$
$$\alpha\,(\tfrac{1}{2}\sin^2\alpha + \tfrac{3}{2}\cos^2\alpha)\}$$
$$+ a\,b\,\text{R} \times \{-\tfrac{3}{4}\sin\alpha\cos\alpha +$$
$$\alpha\,(\tfrac{3}{4}\cos^2\alpha - \tfrac{3}{4}\sin^2\alpha) + \alpha^2\sin\alpha\cos\alpha\}$$
$$+ \frac{\text{W}}{2} \times \{-\cos\alpha + \cos^2\alpha - \tfrac{1}{2}\sin^2\alpha + \alpha\sin\alpha\cos\alpha\}.$$

PROPOSITION VI. *To determine the value of H.*

In this step is embodied the consideration of the lateral action on the abutment. If we omit H in every part of the investigation, we suppose that the ends of the arch rest on the top of flat piers. If we suppose, as above, that the spread is calculated relatively to what it would have been without any strain, and if we then (as we proceed to do) make the spread $= 0$, this implies that the arch is planted in abutments, allowing only the same width as if the arch had been laid flatwise on a floor: now making the spread $= 0$, we obtain

$$\text{H} \times \{\tfrac{3}{2}\sin\alpha.\,\cos\alpha - \alpha\,(\tfrac{1}{2}\sin^2\alpha + \tfrac{3}{2}\cos^2\alpha)\}$$
$$= a\,b\,\text{R} \times \{-\tfrac{9}{4}\sin\alpha.\,\cos\alpha + \alpha\,(\tfrac{9}{4}\cos^2\alpha - \tfrac{1}{4}\sin^2\alpha)$$
$$+ \alpha^2\sin\alpha.\,\cos\alpha\}$$
$$+ \frac{\text{W}}{2} \times \{-\cos\alpha + \cos^2\alpha - \tfrac{1}{2}\sin^2\alpha + \alpha\sin\alpha\cos\alpha\}$$

which gives H in two troublesome fractions. There is no difficulty in expressing it in any numerical instance.

It is worthy of remark here that, though it has been absolutely necessary to use the law and modulus of elasticity or extensibility in the investigation, yet they totally disappear from the result (which does not contain E.)

PROPOSITION VII. *To form the explicit value of M, for every point of the arch.*

In the expression for M, arrived at in Proposition III., the value just indicated for H is to be substituted. It (the value for H) multiplies R $(\cos\theta - \cos\alpha)$.

PROPOSITION VIII. *To find the place where the bending moment is greatest.*

Since a and b are supposed constant, M must be maximum at such place or places. Therefore, differentiating the expression for M, arrived at in Proposition III., and omitting the multiplier R, we get

$$0 = -(a\,b\,\mathrm{R} + \mathrm{H})\sin\theta + a\,b\,\mathrm{R}\,(\sin\theta + \theta\cos\theta) + \frac{\mathrm{W}}{2}\cos\theta,$$

or

$$0 = -\mathrm{H}\sin\theta + a\,b\,\mathrm{R} \times \theta\cos\theta + \frac{\mathrm{W}}{2}\cos\theta.$$

This equation can be solved only in a numerical form. The point or points indicated by it correspond to maxima of the bending moment.

There is, however, another most important breaking point, which the investigation by evanescence of the differential coefficient fails to indicate; and the reason is very remarkable. In forming the value of M (Proposition III.), we have taken the mechanical moments of all the forces on the left of P, namely, of abutment reactions, and of weight of the arch on the left of P. If, in this manner, we investigate the bending moments at successive points, from A towards

W, always taking forces on the left of P, we have still only abutment reactions, and arch on the left of P. But if we continue on the same system of always taking forces on the left of P, as soon as we pass the weight we have the weight W in addition to the abutment reactions and arch on the left of P. Here is a new element introduced. In consequence of this, the function expressing M is discontinuous; $\frac{dM}{d\theta}$ changes *per saltum* at W, and we have no evidence from $\frac{dM}{d\theta}$ whether M is there maximum or not. Usually (not invariably) in such cases, M is maximum or minimum; and it is so here, being maximum negative (as will easily be perceived from the nature of the forces' action), or tending to break the crown downwards with maximum force.

Example:

Let $\alpha = 60^\circ$

The equation in Proposition VI. for the determination of H will be found to give

$$H = a\,b\,R \times .785 + \frac{W}{2} \times 1.261$$

Substituting this value of H in the equation arrived at in Proposition III., we get for the bending moment M at any point, R × the following quantity

$$(a\,b\,R \times 1.785 + \frac{W}{2} \times 1.261)\,(\cos\theta - \tfrac{1}{2}) +$$

$$a\,b\,R\,(\theta \sin\theta - .907) + \frac{W}{2}\,(\sin\theta - .866),$$

and in like manner the equation which determines the place of maximum bending moment (Proposition VIII.) becomes by substitution

$$0 = (-\,a\,b\,R \times .785 - \frac{W}{2} \times 1.261)\sin\theta +$$

$$a\,b\,R \times \theta \cos\theta + \frac{W}{2} \times \cos\theta$$

To proceed further we must assign a numerical value to W, and we shall make three suppositions as follows:

Supposition 1.

Let $W = 0$

The equation for place of greatest moment becomes

$$-.785 \sin \theta + \theta \cos \theta = 0 \text{ or } \tan = \frac{\theta}{.785}$$

The solutions of this are sensibly $\theta = 0° \ \theta = 45°$
Hence the two breaking places are at $\theta = 0° \ \theta = 45°$
The expression for bending moment becomes

$$M = a\, b\, R^2 \times \{1.785 (\cos \theta - \tfrac{1}{2}) + (\theta \sin \theta - .907)\}$$

when $\theta = 0$ this becomes $M = - a\, b\, R^2 \times .014$

when $\theta = 45°$ it becomes $M = + a\, b\, R^2 \times .018$

(The negative sign in the first value of M denotes that the joint opens below).
Hence the weakest place is at 45°.

Supposition 2.

Let $W = R.\, a.\, b.$

(This is nearly, but not quite, the same as supposing that the load on the crown = weight of one-half of the arch.)

One solution for the place of greatest bending moment is $\theta = 0$, the other is given by the equation

$$-1.416. \sin \theta + (\theta + \tfrac{1}{2}) \cos \theta = 0 \text{ or } \tan \theta = \frac{\theta + 0.5}{1.416}$$

the solution of which is $\theta = 41°$ nearly.

The expression for bending moment becomes

$$M = a\, b\, R^2 \times \{2.416 \times \cos \theta + \theta \sin \theta + \tfrac{1}{2} \sin \theta - 2.548\}$$

when $\theta = 0$ this becomes $M = - a\, b\, R^2 \times .142$

when $\theta = 41°$ it becomes $M = + a\, b\, R^2 \times .072$

Hence the weakest place is the crown of the arch. The bending moment at the crown is ten times as great as in Supposition 1: that at the haunch is four times as great.

Supposition 3.

Let $W = 2\,R\,a.\,b.$

(This is nearly the same as supposing that the load on the crown=weight of the whole arch.)

One solution for the place of greatest bending moment is $\theta=0$. The other is given by the equation

$$-\,2.046.\ \sin\theta + (\theta + 1)\cos\theta = 0 \text{ or } \tan\theta = \frac{\theta+1}{2.046}$$

The solution of this equation is $\theta = 39°$ nearly.

The expression for bending moment becomes

$$M = ab\,R^2 \times \{3.046 \times \cos\theta + \theta.\ \sin\theta + \sin\theta - 3.296\}$$

$$\text{when } \theta = 0 \text{ this becomes } M = -\,ab\,R^2 \times .250$$

$$\text{when } \theta = 39° \text{ it becomes } M = +\,ab\,R^2 \times .128$$

Hence the weakest part is the crown.

The bending moment is eighteen times greater than in Supposition (1).

PROPOSITION IX. *To investigate the bending moment at any point of the arch when the weight is eccentric.*

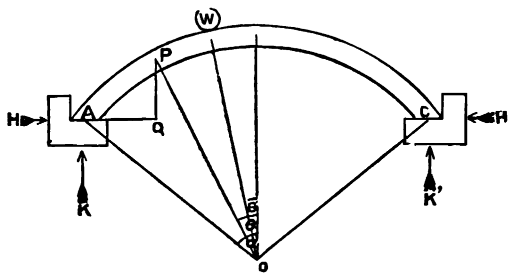


FIG. 4.

Let β be the angle which defines the position of the weight, as in the diagram; then the vertical forces K, K', at the abutments will become respectively

$$R.\ \alpha.\ a\ b + W \times \frac{\sin\alpha + \sin\beta}{2\sin\alpha},$$

and

$$R.\ \alpha.\ a.\ b + W \times \frac{\sin\alpha - \sin\beta}{2\sin\alpha}.$$

The general principles of the investigation will be those of Propositions III. to V., but a peculiar caution is necessary, because the moments produced by the weight W have to be taken differently for points on its two sides. This apparent complexity might be avoided by measuring the moments in opposite directions on the two sides of W. The process will, however, be rendered more intelligible by always proceeding with the arc from the right abutment towards the left, and always taking the moments of parts towards the left.

Section (1). For a point between the right abutment and the weight: that is, for values of θ included between $-\alpha$ and $+\beta$.

The forces on the left hand of the point, producing moments about the point, are

The weight of the portion from the point to A.

The force H.

The weight W acting immediately.

The force $K = R\,\alpha\,a\,b + W \times \dfrac{\sin\alpha + \sin\beta}{2\sin\alpha}$.

The first three of these forces tend to bend A downwards and inwards, and the last to bend A upwards and outwards. They produce (nearly as in Proposition III.) the bending moment.

$$M_1 = \left\{ \begin{array}{l} a.\,b.\,R^2\,(\cos\theta + \theta\sin\theta - \cos\alpha - \alpha.\sin\alpha) \\ +\,H.\,R.\,(\cos\theta - \cos\alpha) \\ +\,W\,R\,(\sin\beta - \sin\theta) \\ +\,W \times \dfrac{\sin\alpha + \sin\beta}{2\sin\alpha} \times R\,(\sin\theta - \sin\alpha) \end{array} \right\}$$

And, as in Propositions IV. and V., this moment, in the space $\delta\theta$, produces a curvature expressed by multiplying it by $E.\,R.\,\delta\theta$: and produces a spread of A expressed by further multiplying it by

$$R\,(\cos\theta - \cos\alpha).$$

Hence, for the spread produced by the whole part between the right abutment and W, we have to integrate $E.\,R.^2\,M_1$ $(\cos\theta - \cos\alpha)\,\delta\theta$ from $\theta = -\alpha$ to $\theta = +\beta$.

Section (2). For a point between the weight and the left abutment; that is, for values of θ included between $+\beta$ and $+\alpha$.

On this section there is no force W to the left of the point: the bending moment is,

$$M_2 = \left\{ \begin{array}{l} a\, b\, R^2 (\cos\theta + \theta . \sin\theta - \cos\alpha - \alpha . \sin\alpha) \\ + H . R (\cos\theta - \cos\alpha) \\ + W . \dfrac{\sin\alpha + \sin\beta}{2\sin\alpha} \times R (\sin\theta - \sin\alpha) \end{array} \right\}$$

and M_2, as before, is to be multiplied by E. R. $\delta\,\theta$, and by R $(\cos\theta - \cos\alpha)$, to find the spread which the bend in the small piece $\delta\,\theta$ produces. Hence for the spread produced by the whole section between the weight and the left abutment, we have to integrate M_2. E. R^2 $(\cos\theta - \cos\alpha)\ \delta\,\theta$ from $\theta = \beta$ to $\theta = \alpha$.

Then, as in Proposition VI., the total spread (the sum of the two integrals) is to be made $= 0$; H will be determined from this equation, and its value must be substituted in the expressions for M_1 and M_2. The bending moment at any point of either section of the arch will then be obtained; it will have one maximum value where $\theta = \beta$, and one other maximum value in each section of the arch where $\dfrac{dM}{d\theta} = 0$.

The total spread of the foot A of the arch is thus obtained. The three integrals in Section (2) are similar to the 1st, 2nd, and 4th integrals in Section (1); the latter are to be taken between limits $-\alpha$ and $+\beta$, and the former between limits $+\beta$ and $+\alpha$. The sum of the two parts of each integral, therefore, constitute integrals to be taken between limits $-\alpha$ and $+\alpha$. The result of the integration will give (omitting the general factor E. R^3),

$$\begin{array}{c} (a\, b\, R + H) \times \{ -3 \sin\alpha \cos\alpha + \\ \alpha (\sin^2\alpha + 3\cos^2\alpha) \} \\ + a\, b\, R \times \{ -\tfrac{3}{2} \sin\alpha\cos\alpha + \\ \alpha (\tfrac{3}{2}\cos^2\alpha - \tfrac{3}{2}\sin^2\alpha) + \alpha^2\, 2 \sin\alpha\cos\alpha \} \end{array}$$

$$+ \mathrm{W} \frac{\sin \alpha + \sin \beta}{2 \sin \alpha} \times (- 2 \sin^2 \alpha + 2 \sin \alpha \cos \alpha).$$

The 3rd integral in Section (1) is to be taken only between the limits $-\alpha$ and $+\beta$. It will be found to give a result,

$$\mathrm{W} \times \left\{\tfrac{1}{2} + \tfrac{1}{2} \sin^2 \beta + \tfrac{1}{2} \cos^2 \alpha - \cos \alpha . \cos \beta + \sin \alpha \sin \beta - (\alpha + \beta) \cos \alpha \sin \beta\right\}$$

And collecting all the terms, we obtain for the entire inwards spread,

$$\mathrm{E\,R}^3 \left\{ \begin{array}{l} \mathrm{H} \times \{ -3 \sin \alpha . \cos \alpha + \alpha (\sin^2 \alpha + 3 \cos^2 \alpha)\} \\ + a\, b\, \mathrm{R} \times \{- \frac{9}{2} \sin \alpha . \cos \alpha + \alpha (- \frac{1}{2} \sin^2 \alpha \\ \quad + \frac{9}{2} \cos^2 \alpha) + \alpha .^2\, 2 \sin \alpha \cos \alpha\} \\ + \mathrm{W} \times \{\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \cos^2 \beta - \cos \alpha . \cos \beta + \\ \quad (\alpha . \sin \alpha - \beta . \sin \beta). \cos \alpha\} \end{array} \right\}$$

(The following partial verifications are to be found on comparing this formula with that obtained in Proposition V.: (1) When $\beta = 0$, its value is double that of the spread in Proposition V.; (2) The powers of (β) are even, as will be seen by expanding the functions of β; (3) When $\beta = \alpha$, the coefficient of W vanishes.)

Example :

Let $\alpha = 60^\circ$ $\qquad \beta = 30^\circ$

Then substituting these values of (α) and (β) in the last equation we obtain,

$$\text{Spread of } \mathrm{A} = \mathrm{E.R}^3 \times \{\mathrm{H} \times .2718 - a\, b\, \mathrm{R} \times .2134 - \mathrm{W} \times .1104\}$$

and putting this$=0$, we obtain

$$\mathrm{H} = a\, b\, \mathrm{R} \times .785 + \mathrm{W} \times .406$$

The absolute bending moment M_1, for any point between $\theta = -\alpha$ and $\theta = +\beta$, is found by substituting this value of H in the expression for M_1, and we shall find,

$$\mathrm{M}_1 = \left\{ \begin{array}{l} a\, b\, \mathrm{R} \times \{1.785 \cos \theta + \theta . \sin \theta - 1.800\} \\ + \mathrm{W} \{.406 \cos \theta - .211 \sin \theta - 0.386\} \end{array} \right\}$$

The absolute bending moment, M_2, for any point between $\theta = +\beta$ and $= +\alpha$ will in the same way be found to be as follows:

$$M_2 = \begin{Bmatrix} a\,b\,R\,(1.785 \cos\theta + \theta.\sin\theta - 1.800) \\ + W\,(.406 \cos\theta + .789 \sin\theta - 0.886) \end{Bmatrix}$$

The two formulæ agree when $\theta = 30^\circ$ or under the weight W. We shall give no further attention to the second, as that side of the arch is the stronger.

To find where M_1 is a maximum, we must make $\frac{d\,M_1}{d\,\theta} = 0$

This will give us,

$$a\,b\,R \times \{-1.785 \sin\theta + \sin\theta + \theta.\cos\theta\} + W\{-.406 \sin\theta - .211 \cos\theta\} = 0$$

from whence we obtain

$$\tan\theta = \frac{a\,b\,R \times \theta - W \times .211}{a\,b\,R \times .785 + W \times .406}.$$

$$\text{Let } W = R\,a\,b. \quad \text{Then } \tan\theta = \frac{\theta - .211}{1.191}.$$

This gives $\theta = -34^\circ$ nearly. Substituting in the expression for M_1 above, we get,

when $\theta = +30^\circ$ $\quad M_1 = -.133 \times a\,b\,R^2$

when $\theta = -34^\circ$ $\quad M_1 = +.080 \times a\,b\,R^2$

Therefore the bending moment is greatest for the point under the weight.

$$\text{Let } W = 2\,R\,a\,b. \quad \text{Then } \tan\theta = \frac{\theta - .422}{1.597}.$$

This gives $\theta = -31^\circ$, and as before we get,

when $\theta = +30^\circ$ $\quad M_1 = -.274 \times a\,b\,R^2$

when $\theta = -31^\circ$ $\quad M_1 = +.148 \times a\,b\,R^2$

Thus in this case the bending moment is nearly twice as great for the point under the weight as for any other point.

Proposition X. *To investigate the bending moment at any point of the arch, when the piers present no lateral resistance.*

In this case we have merely to make $H=0$ in the expressions for M, and to enter into none of the calculations for spread. And in the numerical calculation it is only necessary to consider the point under the weight; as it is certain that, if W be not very small, the arch will break there.

If the weight be central, the value of M (Prop. III.) is $R \times$ the following expression:

$$a\,b\,R \times \{ (\cos\theta - \cos\alpha) + (\theta \sin\theta - \alpha \sin\alpha) \}$$
$$+ \frac{W}{2} (\sin\theta - \sin\alpha)$$

when $\theta = 0$ this becomes

$$M = a\,b\,R^2 \times (1 - \cos\alpha - \alpha \sin\alpha) - R\frac{W}{2} \sin\alpha$$

and if $\alpha = 60^\circ$, this becomes

$$M = -R\,(a\,b\,R \times .407 + W \times .433)$$

Let $W = 0$ then $M = -a\,b\,R^2 \times .407$

Let $W = R\,a\,b$ then $M = -a\,b\,R^2 \times .840$

Let $W = {}_2R\,a\,b$ then $M = -a\,b\,R^2 \times 1.273$

If the weight be eccentric, the value of M_1 (Prop. IX.) is $R \times$ the following expression:

$$\left\{ \begin{array}{l} a\,b\,R \times (\cos\theta + \theta \sin\theta - \cos\alpha - \alpha \sin\alpha) \\ + W (\sin\beta - \sin\theta) + W.\, \dfrac{\sin\alpha + \sin\beta}{2 \sin\alpha} \times \\ (\sin\theta - \sin\alpha) \end{array} \right\}$$

and when $\theta = \beta$ this becomes

$$M = a\,b\,R^2 \times (\cos\beta + \beta.\, \sin\beta - \cos\alpha - \alpha \sin\alpha)$$
$$- W.\,R.\, \frac{\sin^2\alpha - \sin^2\beta}{2 \sin\alpha}.$$

And if $\alpha = 60^\circ$ and $\beta = 30^\circ$ this becomes

$$M = -R\,\{a\,b\,R \times .279 - W \times .289\}.$$

Let $W = R\,a\,b$ then $M = -a\,b\,R^2 \times .568$

Let $W = 2\,R\,a\,b$ then $M = -a\,b\,R^2 \times .857.$

Table of Results collected from Propositions VIII., IX., and X.

In the following formulæ the section of the arched rib is supposed to be a parallelogram of depth (b) and width (a), R is the radius of the arch at the middle of its depth. The whole arch is supposed to be 120° of a circle, or 60° on each side of the crown. When the weight W is eccentric, it is supposed to be 30° from the crown, or midway between the crown and the springing. M is the bending moment at the point specified.

PART I.—*The arch tight within the abutments, so that they receive the full horizontal thrust.*

(1)	No weight W	
	At the crown	$M = -a\,b\,R^2 \times .014$
	At 45° from the crown	$M = +a\,b\,R^2 \times .018$
(2)	The weight central	$W = R\,a\,b$
	At the crown	$M = -a\,b\,R^2 \times .142$
	At 41° from the crown	$M = +a\,b\,R^2 \times .072$
(3)	The weight central	$W = 2\,R\,a\,b$
	At the crown	$M = -a\,b\,R^2 \times .250$
	At 39° from the crown	$M = +a\,b\,R^2 \times .128$
(4)	The weight eccentric	$W = R\,a\,b$
	Under the weight	$M = -a\,b\,R^2 \times .133$
	At 34° from the crown on the opposite side	$M = +a\,b\,R^2 \times .080$
(5)	The weight eccentric	$W = 2\,R\,a\,b$
	Under the weight	$M = -a\,b\,R^2 \times .274$
	At 31° from the crown on the opposite side	$M = +a\,b\,R^2 \times .148$

PART II.—*The arch not confined horizontally between the abutments.*

(1)	No weight W	
	At the crown	$M = -a\,b\,R^2 \times .407$

(2)	The weight central	$W = R\,a\,b$
	At the crown	$M = -\,a\,b\,R^2 \times .840$
(3)	The weight central	$W = 2\,R\,a\,b$
	At the crown	$M = -\,a\,b\,R^2 \times 1.273$
(4)	The weight eccentric	$W = R\,a\,b$
	Under the weight	$M = -\,a\,b\,R^2 \times .568$
(5)	The weight eccentric	$W = 2\,R\,a\,b$
	Under the weight	$M = -\,a\,b\,R^2 \times .857$

It will now be proper to apply the formulæ of the foregoing investigation to the circumstances of a practical example in order to show the importance of the bending moment when combined with the thrust force of the arch. The problem will be simplified as much as possible, and may be stated as follows:

Required the sectional dimensions for a continuous wrought-iron arch of 200 ft. span (measured from centre to centre of the feet of the arch), and 30 ft. mean central rise: the section of the arch to have the form of a single rectangular cell of uniform thickness: the fixed load (comprising the weight of the arch itself and its proportion of the roadway) to be taken at 2 tons per foot run, and the running load at $1\frac{1}{4}$ tons per foot run?

With the above dimensions it will be found that the radius $R = 181.666$ ft., the half angle of the arch, $\alpha = 33.\ 24'$ (circular measure of $\alpha = .5829$), and length of arch $= 211.8$ ft.

We shall treat the case as if the arch were loaded uniformly all over its length: this supposition, though incorrect, will give us a close approximation to the circumstances of the actual case. It is usual to make the strengths sufficient to carry 3 times the fixed load + 6 times the running load. Hence in the case of the example the load per foot run will be $6 + 7\frac{1}{2} = 13\frac{1}{2}$ tons per foot run, or 2700 tons in the aggregate:

this load distributed over the arch will give a uniformly distributed load of $\frac{2700}{211.8} = 12.75$ tons per foot of the arch.

We shall proceed to determine the strains from consideration of an arch having the same shape and size as the example, and of such sectional dimensions as to weigh 12.75 tons per foot of its length; taking 480 lb. as the weight of 1 cubic foot of wrought iron, the sectional area must be $59\frac{1}{2}$ square feet. It will be understood that the value of the sectional area is merely introduced in this form to suit the unit of weight (1 cubic foot of iron) adopted in the preceding investigation, and as representing in such units the actual load of 12.75 tons per foot run.

(1.) To find the value of H, which will be the value of the thrust force at the crown of the arch.

Taking the formula for H, given in Proposition VI., and putting $W=0$, as there is no weight, W, in the present example,

$$H \times \{\tfrac{3}{2} \sin \alpha . \cos \alpha - \alpha (\tfrac{1}{2} \sin^2 \alpha + \tfrac{3}{2} \cos^2 \alpha)\} =$$
$$a\, b\, R \times \{- \tfrac{9}{4} \sin \alpha . \cos \alpha + \alpha (\tfrac{9}{4} \cos^2 \alpha - \tfrac{1}{4} \sin^2 \alpha) + \alpha^2 \sin \alpha . \cos \alpha\}$$

and putting $\alpha = 33.^\circ\, 24'$, the above equation will be found to give a result.

$$H \times .0085 = a\, b\, R \times .0079 \text{ or } H = .93 \times a\, b\, R.$$

(2.) To find the place of maximum strain. The equation is found in Proposition VIII.

$$0 = - H . \sin \theta + a\, b\, R \times \theta \cos \theta.$$
$$\text{or, } 0 = - .93 \times \sin \theta + \theta \cos \theta.$$

This gives $\tan \theta = \frac{\theta}{.93}$, and the solution is found by trial to be $\theta = 26^\circ$; $\theta = 0$ will also define a point of maximum strain.

(3.) To find the bending moment, and the thrust force, at the points defined by $\theta=0$, $\theta=26^\circ$.

The equation for the bending moment is found in Proposition III.; it becomes, putting $W=0$, and substituting the value of H,

$$M = a\,b\,R^2 \times 1.93 \times (\cos\theta - \cos\alpha) + a\,b\,R^2 \times (\theta\sin\theta - \alpha\sin\alpha),$$

when $\theta = 0$ and $\alpha = 33^\circ\ 24'$. $M = -.0023 \times a\,b\,R^2$

when $\theta = 26^\circ$ and $\alpha = 33^\circ\ 24'$. $M = +.0054 \times a\,b\,R^2$

Thus the bending moment is greatest at the point $\theta = 26^\circ$, and its numerical value at that point is

$$M = .0054 \times a\,b\,R^2 = .0054 \times 59\tfrac{1}{2} \times (181.66)^2 = 10603.$$

The thrust force at the crown is $H = .93 \times a\,b\,R$, and the thrust force at the point $\theta = 26^\circ$, will be $H \times \sec 26^\circ$.

$$= .93 \times a\,b\,R \times 1.112 = 11177.$$

(The units being the foot, and the weight of a cubic foot of wrought iron.)

(4.) To determine the dimensions of the section so as to bear the strain due to the thrust force and the bending moment without exceeding a maximum thrust of 12 tons per square inch at the compressed surface of the arch.

The best way to do this will be by trial, thus: We shall assume the depth and width of the section, and determine the thickness of the shell, so that the sectional area shall be greater than would be necessary to endure the thrust force only. Thus we shall assume the thickness in our example, so as to give a sectional area of $\frac{2391}{7}$ square inches (the thrust force=11,177 cubic feet=2391 tons), instead of $\frac{2391}{12}$ square inches, which would be about the least that would resist the thrust force only. We shall then try if this section will endure the bending moment in addition without producing a greater additional strain at the parts furthest from the neutral line (on the compressed side of the arch) than 5 tons per square inch, which, together with the 7 tons already assumed to be caused by the thrust force only, will bring the metal at the outside of the section to its limiting endurance of 12 tons per square inch. This will be the first trial, and if the section will not resist the bending moment in

addition to the thrust force, we must make a second trial with altered assumptions.

Let, then, the depth of the rib be 6 ft., and the breadth 4 ft., and let the uniform thickness be k; then the sectional area will be $20 \times 12 \times k = 240.k$ square inches nearly. Now the thrust force $= 2391$ tons, and the sectional area required to resist this force, allowing 7 tons per square inch, is $\frac{2391}{7} = 342$ square inches, nearly. Therefore, $240 \times k = 342$, and $k = 1.425$ in. $= .11875$ ft.

Now, the moment of the forces resisting the bending moment, in the case of a cell section such as that assumed is:

$$\frac{2\,t}{3\,f}\left[a\ \left\{f^3 - (f-k)^3\right\} + 2\,k\,(f-k)^3\right]$$

Where t = strain of the metal at the outside of the arch on the compressed side, and f = distance of the neutral axis from the same side of the arch. In the present case, the section being symmetrical, we have f = half depth of rib $= 3$ ft.; also $a = 4$ ft., and $k = .11875$ ft. The above expression becomes $= 4.t$ very nearly. If we put this equal to the bending moment, we obtain $4.t = 10603$, and

$t = 2651$ cubic feet of iron per square foot of section.
$= 568$ tons „ „ „
$= 3.94$ „ „ square inch „

Thus, the total strain on the metal at the outside surface of the arch (where it is most strained) is $7 + 3.94 = 10.94$ tons per square inch, which does not exceed the limiting strength of 12 tons. The arch is therefore safe, and the thickness of the shell might with safety be somewhat reduced.

It is worth while to notice by this example how exceedingly fallacious a result would be obtained, if the bending moment of the forces were neglected. If the arch had formed a larger segment of a circle, as, for instance, if it had been an arch of 120° instead of 67°, the bending moment would have been far more serious, and, in such cases, a fairly approximate